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Note

Implicit Filtering in Conjunction with Explicit Filtering

1. Introduction

In the solution of nonlinear systems of equations spurious growth of short waves, especially $2\Delta x$, takes place frequently (see, for example, Phillips, [1]). A common way of overcoming this noise is by filtering out the short waves. The most basic filter for doing this is defined by:

$$\bar{\phi}_j = \phi_j + S(\phi_{j-1} + \phi_{j+1} - 2\phi_j), \tag{1}$$

where S is a smoothing element; $j \pm 1 \to (j \pm 1)\,\Delta x$; Δx is a space increment.

For $S = \frac{1}{4}$, the filter (1) removes thoroughly the $2\Delta x$ wave and damps all the longer waves. But for most meteorological purposes the smoothing of the longer waves is too strong. In order to prevent this undesired smoothing, Shapiro [2] constructed the n-element smoothing operator which consists of n basic operators in form (1) with the following essential properties:

(1) Removal of waves of 2 grid intervals

and

(2) Damping of all other waves. The damping may be decreased to the minimum desired level by choosing a large enough n.

For the same purpose Long *et al.* [3] suggested a very selective low-pass filter in the form

$$(1-\delta)\,\bar{\phi}_{j-1} + 2(1+\delta)\,\bar{\phi}_j + (1-\delta)\,\bar{\phi}_{j+1} = \phi_{j-1} + 2\phi_j + \phi_{j+1}, \tag{2}$$

where ϕ is the field to be smoothed and $\bar{\phi}$ the smoothed field. This implicit filter completely eliminates the $2\Delta x$ waves with each application, while its smoothing effect on the longer wavelengths is a function of δ. Since then it has been adopted successfully by some authors. Mahrer and Pielke [4] used it in modelling the sea breeze and mountain flow. Pepper *et al.* [5] modeled atmospheric pollution, while Kemper *et al.* [6] reported good results with the implicit filter. We also adopted this filter in a mesometeorological model with topography developed at the Hebrew University of Jerusalem [7]. The filter's properties were discussed briefly by the above-mentioned authors and investigated by Long. In this note we are interested mainly in comparing the implicit filter with the explicit one.

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2. Comparison Between the Implicit Filter and the Ideal Explicit Filter Suggested by Shapiro

The response functions of the implicit filter (2) may be shown (through substitution of the eigenfunction $\phi_j = e^{i\lambda j}$) to be:

$$R(\delta, \lambda) = \bar{\phi}/\phi = \frac{1}{1 + \delta \operatorname{tg}^2(\lambda/2)}, \tag{3}$$

where λ is the wavenumber. Now, the following features of the implicit filter are immediately obvious: for $\delta = 0$ the filtered and unfiltered values are equal; as $\delta \to 0^+$ the $2\Delta x$ waves are eliminated while longer waves are not damped; for $\delta = 1$ the filter becomes the basic smoother (1) with $S = \frac{1}{4}$.

The response function for the n-smoothing operator suggested by Shapiro [2]–his formula (24)—is:

$$\rho_n(\lambda) = 1 - \sin^{2n}(\lambda/2), \tag{4}$$

where $\lambda = k\,\Delta x$.

In order to compare that filter to the implicit filter let us equate both response functions, i.e., (3) and (4). In atmospheric dynamic problems we are interested in minimum smoothing of the longer waves, i.e., very small δ, thus we may assume $\delta \operatorname{tg}^2(\lambda/2) \ll 1$ (excluding the case where $\lambda = \pi$, i.e., the $2\Delta x$ wave) then for the implicit filter

$$R(\delta, \lambda) = 1 - \delta \operatorname{tg}^2 \frac{\lambda}{2} + \delta^2 \operatorname{tg}^4 \frac{\lambda}{2} - \cdots \approx 1 - \delta \operatorname{tg}^2 \frac{\lambda}{2}. \tag{5}$$

This approximation is better as long as δ is smaller. Comparison of (5) and (4) yields

$$\delta_{\lambda_0} = \cos^2 \frac{\lambda_0}{2} \sin^{2(n-1)} \frac{\lambda_0}{2}; \tag{6}$$

i.e., the implicit filter (2) with $\delta = \delta_{\lambda_0}$ smooths the wave denoted by $\lambda_0 = k_0\,\Delta x$, approximately the same as an explicit filter of order n, k_0 being the wavenumber corresponding to λ_0.

Formula (6) enables us to choose such a δ for the implicit filter that the response function for any desired wavenumber (excluding the wave $2\Delta x$) will equal that for an "ideal" explicit operator with order n. Moreover, in Appendix I we prove that the response function for the implicit filter with $\delta = \delta_{\lambda_0}$ is greater for all waves between $2\Delta x$ to wavenumber k_0, and smaller for longer waves (relative to λ_0).

It turns out that we may choose λ_0 large enough in accordance with our problem (and the grid) such that for the largest given n there is always δ_{λ_0} small enough so that the implicit filter suitable is better (more "ideal"?) than the "ideal" explicit filter. Let us visualize how easy this procedure is in the following example:

EXAMPLE:

$$n = 2 \text{ and wavelength } L = 10\Delta x;$$

i.e.,

$$\text{wavenumber } k_0 = \pi/(5\Delta x), \qquad \lambda_0 = \pi/5;$$

then, according to (6)

$$\delta_{\lambda_0} = \cos^2\frac{\lambda_0}{2}\sin^2\frac{\lambda_0}{2} = \frac{1}{4}\sin^2\lambda_0 \approx 0.08. \tag{7}$$

To determine how good approximation (7) is we check by direct application of both response functions (3)—with δ_{λ_0}—and (4) for $L = 10\Delta x$:

$$R(L = 10\Delta x) = 0.9909;$$

$$\rho_2(L = 10\Delta x) = 0.9908.$$

Thus, (5) is a reasonable approximation in this case.

Table I shows the response of both functions for this example as a function of wavelength for a single application.

It is important to note that for long waves ($\lambda \to 0$) the explicit filter damps less, i.e., $\rho_n > R$ (for the implicit filter $R \to 1 - O(\lambda^2)$, whereas for the Shapiro filter $\rho_n \to 1 - O(\lambda^{2n})$ [Long, personal communication]). It should be stressed, however, that we may choose that point λ_0 below which the reverse relation, i.e., $R > \rho_n$, exists and the implicit filter is less damping. These results are illustrated in Fig. 1 for the above example. Note that point $\lambda_0 \to L = 10\Delta x$ for which the relations are reversed.

Shapiro [2, p. 367] indicates four criteria for the "ideal filter": (1) removing waves of 2-grid intervals; (2) damping all other waves; (3) not altering the average value of the function; and (4) not changing the phase of any component. Since to our mind the implicit filter does as well as, if not better than, that of Shapiro, we find it preferable to Shapiro's ideal filter.

TABLE I

The Response Functions of the "Ideal Filter" and the Implicit Filter as a Function of Wavelength[a]

Number of waves	2	3	4	6	8	10	15	120	50
Implicit filter R	0	0.795	0.921	0.972	0.985	0.990	0.9961	0.9978	0.9996
Explicit filter ρ_2	0	0.437	0.750	0.937	0.978	0.990	0.9981	0.9994	0.9999

[a] The implicit filter was chosen through criterion (12), i.e., $\delta = 0.08$, so that for $10\Delta x$ the response functions are approximately identical.

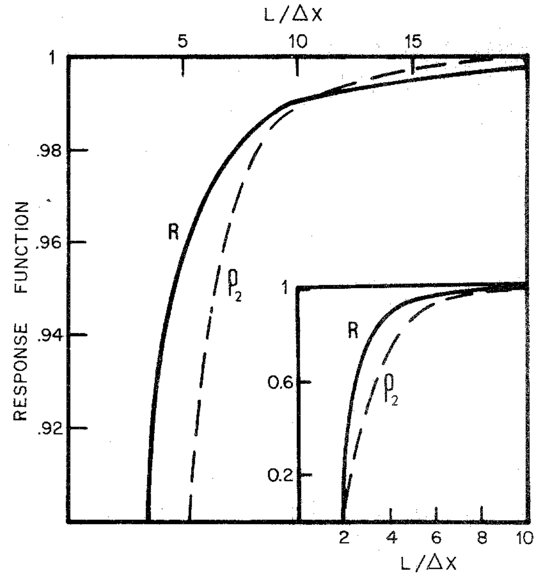

FIG. 1. Response functions of the implicit filter (solid line) and the "ideal filter" of order $n = 2$ (broken line) in two different scales. The illustration in the lower right-hand corner shows the response functions for wavelength interval 2–10Δx. The larger illustration shows the response functions up to wavelength 20Δx.

Following the comparison of the response functions for the implicit filter and the explicit filter of order n it is of interest to compare the filters themselves. To be able to do so, let us introduce the explicit filter which is completely equivalent to the implicit one. As suggested by Long (personal communication) this filter should be infinite, symmetric, and two-sided and thus may be written as

$$\bar{\phi}_j = w_0 \phi_j + \sum_{l=1}^{\infty} w_l (\phi_{j+l} + \phi_{j-l}). \tag{8}$$

The exact expressions for the weights of this explicit equivalent filter are derived in Appendix II; it turns out that w_k creates a geometrical series (starting with w_1) which drops off with the factor $-w_0(1 - \sqrt{\delta})$ and reverses sign. Returning to Shapiro's filter of order n, we find that it may be written with the same formula, i.e., (8), except that the summation is only up to 2^{n-1} (see Shapiro's formula (39)). Now, the following origin to the sharper[1] response possible with the implicit filter, is suggested: for Shapiro's filter to be sharper, higher-order n is needed leading to $(2^n + 1)$ points for the filtration of each value, whereas for the implicit filter an infinite number of points are involved (when boundaries are far at infinity). Anyhow, solution of the necessary tridiagonal system of linear equations required for the implicit implementation is

[1] Sharper response means response function which increases faster to 1 for wavelengths longer than $2\Delta x$.

reasonably efficient, if performed by triangular factorization and back-substitution. Thus, when comparing the relative efficiency, it comes out that for a sharper response it is more costly[2] to use the explicit filter.

Another advantage of the implicit filters appears to be the relative convenience in the process of changing the sharpness of the filter. Making sharper the implicit filter means simply reducing δ. But, to use a sharper explicit filter (higher order n) more points, i.e. $(2^n + 1)$, become involved in the calculation of each smoothed value leading to further consideration of the boundary conditions necessary.

3. Summary and Concluding Remarks

We compared two filters usually used for the same purpose which is mainly removing two-grid interval waves and minimum damping of all longer waves. It is shown that the damping of short waves caused by the implicit filter is less in comparison to the explicit filter. Further, it is shown that the upper limit defining short waves (i.e. λ_0) might be chosen sufficiently high (small λ_0) that for all the waves we are interested in, less damping would occur.

Finally, it is suggested that the relative sharper implicit filter may be explained through the wider string of points involved with the filtration of each value. The exact expressions of the explicit filter equivalent to the implicit one are derived and it is shown that an infinite string of neighbouring points are involved while in the Shapiro's filter of order n only $(2n + 1)$ neighbouring points participate in the filtration of each value. At the same time, it may easily be shown that the implicit filter is more efficient relative to Shapiro's explicit filter with order $n \gtrsim 2$.

Appendix I

Let us prove that

$$R(\delta_{\lambda_0}, \lambda_0) > \rho_n(\lambda) \qquad \text{at} \quad \pi > \lambda \gtrsim \lambda_0 \text{ (i.e., } 2 < m < m_0) \text{ and } 0 < \delta < 1,\ n \geqslant 2,$$

where

$$m = L/\Delta x, \qquad \delta_{\lambda_0} = \cos^2 \frac{\lambda_0}{2} \sin^{2(n-1)} \frac{\lambda_0}{2},$$

$$\lambda_0 = k_0\, \Delta x = 2\pi/m_0,$$

$$R(\delta, \lambda) = \left(1 + \delta \operatorname{tg}^2 \frac{\lambda}{2}\right)^{-1} \approx 1 - \delta \operatorname{tg}^2 \frac{\lambda}{2}, \qquad \text{for} \quad \delta \ll 1, m \geqslant 3,$$

$$\rho_n(\lambda) = 1 - \sin^{2n} \frac{\lambda}{2}.$$

[2] Only Shapiro's explicit filter of order $n = 1$ is cheaper in use relative to the implicit one; the latter being comparable in efficiency with Shapiro's filter of order $n = 2$ (5 point involved).

Proof.

$$m < m_0 \Rightarrow \pi/m > \pi/m_0, \text{ but } m > 2,$$

thus

$$\sin\frac{\pi}{m} > \sin\frac{\pi}{m_0} \quad \text{and} \quad \cos\frac{\pi}{m} < \cos\frac{\pi}{m_0} \Rightarrow$$

$$\frac{\sin \pi/m}{\sin \pi/m_0} > 1, \qquad \frac{\cos \pi/m_0}{\cos \pi/m} > 1.$$

(a) If $m > 4$, then $\sin 2\pi/m_0 < \sin 2\pi/m \Rightarrow \frac{1}{4}\sin^2 2\pi/m_0 < \frac{1}{4}\sin^2 2\pi/m$ or $\cos^2 \pi/m_0 \sin^2 \pi/m_0 < \cos^2 \pi/m \sin^2 \pi/m \Rightarrow$

$$\left[\frac{\cos \pi/m_0}{\cos \pi/m}\right]^2 < \left[\frac{\sin \pi/m}{\sin \pi/m_0}\right]^2$$

but

$$\frac{\sin \pi/m}{\sin \pi/m_0} > 1 \Rightarrow \left[\frac{\cos \pi/m_0}{\cos \pi/m}\right]^2 < \left[\frac{\sin \pi/m}{\sin \pi/m_0}\right]^{2n-2} \qquad \text{for} \quad n \geqslant 2.$$

But

$$\pi/m_0 = \lambda_0/2, \qquad \pi/m = \lambda/2,$$

$$\cos^2\frac{\lambda_0}{2}\sin^{2(n-1)}\frac{\lambda_0}{2}\,\mathrm{tg}^2\frac{\lambda}{2} < \sin^{2n}\frac{\lambda}{2},$$

or

$$\delta_{\lambda_0}\,\mathrm{tg}^2\frac{\lambda}{2} < \sin^{2n}\frac{\lambda}{2} \Rightarrow R(\delta,\lambda) = 1 - \delta_{\lambda_0}\,\mathrm{tg}^2\frac{\lambda}{2} > 1 - \sin^{2n}\frac{\lambda}{2} = \rho_n(\lambda)$$

$$\Rightarrow R(\delta,\lambda) > \rho_n(\lambda).$$

(b) If $2 < m \leqslant 4$ the approximation $\delta\,\mathrm{tg}^2\,\lambda/2 \ll 1$ is no longer valid, but it is easy to prove numerically for $m = 3$, $m_0 \geqslant 4$, and $m = 4$, $m_0 \geqslant 5$ so that the inequality still holds.

APPENDIX II. Derivation of Expressions for w_0, w_l

From (8) and (3),

$$\bar{\phi}_j = w_0\phi_j + \sum_{l=1}^{\infty} w_l(\phi_{j+l} + \phi_{j-l}),$$

$$\bar{\phi}_j = \left(1 + \delta\,\mathrm{tg}^2\frac{\lambda}{2}\right)^{-1}\phi_j.$$

Considering the Fourier component λ, $\phi_j = e^{i\lambda j}$, then by equating the last two equations we get

$$\left[1 + \delta \operatorname{tg}^2 \frac{\lambda}{2}\right]^{-1} = w_0 + 2 \sum_{l=1}^{\infty} w_l \cos l\lambda.$$

Multiplying by $\cos i\lambda$—i an integer—and integrating over all Fourier components leads to the following integral relations for w_0 and w_i:

$$w_0 = \frac{1}{\pi} \int_0^{\pi} \frac{1}{1 + \delta \operatorname{tg}^2 \lambda/2} d\lambda, \qquad w_i = \frac{1}{\pi} \int_0^{\pi} \frac{\cos i\lambda}{1 + \delta \operatorname{tg}^2 \lambda/2} d\lambda.$$

Upon substituting the following identities:

$$\left(1 + \delta \operatorname{tg}^2 \frac{\lambda}{2}\right)^{-1} \equiv \frac{1 + \cos \lambda}{(1 + \delta)(1 + a \cos \lambda)}; \qquad a = (1 - \delta)/(1 + \delta),$$

$$\cos \lambda \cos i = 0.5(\cos(i + 1)\lambda - \cos(i - 1)\lambda),$$

the integrals can be evaluated (see [8]) to obtain:

$$w_0 = \frac{1 - \delta^{1/2}}{1 - \delta}, \qquad w_i = (-1)^{i+1} \frac{\delta^{1/2}}{1 - \delta} \left[\frac{(1 - \delta^{1/2})}{1 - \delta}\right]^i, \qquad i = 1, 2, 3 \ldots.$$

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